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The Effect of Gravity and the Change of Shape

by

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Institute for Mathematics and Mechanics

This Paper Represents Results Obtained at the Institute for Mathematics and Mechanics, New York University, under the Auspices of the Office of Naval Research, Contract No. Nonr-235(02)

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I Introduction

The classical theory of an underwater explosion bubble is based on the assumption that the bubble remains spherical at all times. However actual bubbles do not remain spherical although they may be spherical initially. Instead they become flattened or even kidney-shaped and often break up. This change of shape is mainly due to gravity. The present report presents a theory which describes the change of shape as well as the rise of the bubble, by taking account of gravity.

Several other authors, such as Herring and Ward, have also given theories of the change of bubble shape due to gravity. The present work is intended to be more systematic and more complete than any of the former theories.

In the present theory it is assumed that the water is incompressible and unbounded, and the pressure is assumed to be uniform throughout the bubble at all times. Initially the bubble is assumed to be spherical. Then if we neglect gravity we find that the bubble remains spherical but performs periodic radial oscillations, exactly as in the classical theory. Taking account of first order terms due to gravity, we find that the bubble remains spherical but also rises exactly as predicted by Herring's formula. Higher order terms lead to the change of shape and sometimes to the break-up of the bubble, and also lead to a modification of the Herring rise formula. All of these results are discussed in section VIII of this report.

The assumption that the water is incompressible can be removed, as it was in our previous report on the spherical bubble. However it does not seem worthwhile to add the complication of compressibility to the other difficulties of the present problem. Furthermore in section VIII we have indicated how the main effects of compressibility can be obtained by combining the present results with those of our previous report.

The assumption that the water is unbounded can also be removed, and the theory without this assumption is the subject of our next report. In that report it will be shown that the results of the present report provide a first approximation to the solution when the bubble is not too near the boundaries. Of course further approximations will also be given.

In section II the problem is formulated, in section III moving coordinates are introduced and in section IV dimensionless variables are defined. The method of solution is explained in section V and the solution up to and including the third order is considered in section VI. A special case is treated explicitly in section VII. Finally in section VIII all the results are discussed and some general conclusions are drawn.

II Formulation

We assume that an incompressible invistid fluid of infinite extent contains a gas bubble within it. The velocity $u(x,y,z,t)$ of the fluid is assumed to be derivable from a potential function $\phi(x,y,z,t)$ which satisfies Laplace's equation

(1) $\vec{u} = \nabla\phi$

(2) $\nabla^2\phi = 0$

The pressure $p(x,y,z,t)$ in the water is then given by the Bernoulli equation

(3) $p = m(t) - \rho\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right] - \rho g z$

In (3) ρ is the density of water, g is the acceleration of gravity and $m(t)$ is an arbitrary function of time. We have also assumed that the positive z axis points vertically upward.

The velocity $\nabla\phi$ is assumed to vanish at infinite distance from the bubble

(4) $\nabla\phi \rightarrow 0$ as $(x^2 + y^2 + z^2)^{1/2} \rightarrow \infty$.

From (1) it is clear that ϕ is defined up to an additive function of t. This indefiniteness can be removed by specifying ϕ or ϕ_t at one point for all t. For this purpose we assume

(5) $\phi \rightarrow 0$ as $(x^2 + y^2 + z^2)^{1/2} \rightarrow \infty$.

The condition (5) implies that $\phi_t \rightarrow 0$ at infinity, and also implies (4), which can therefore be omitted.

The arbitrary function $m(t)$ in (3) can be determined by specifying p at one point for all t. For this purpose we assume that

(6) $p(x,y,o,t) \rightarrow P_o$ as $(x^2 + y^2) \rightarrow \infty$

Thus $m(t)$ can be determined from (3,5,6) which yield

(7) $m(t) = P_o$.

We call P_o the pressure at infinity at z=0. [It is just the hydrostatic pressure at the level z=0, namely $P_o = p_o + \rho g z_o$, if p_o is the atmospheric pressure and $z_o > o$ is the depth of z=o below the water surface.]

The bubble surface is assumed to be given by the equation

(8) $F(x,y,z,t) = 0$

Since the normal velocity of the bubble surface must be the same as that of the adjacent water, F must satisfy the kinematic condition

(9) $\nabla F \cdot \nabla\phi + F_t = 0$ on $F = 0$.

The pressure π within the bubble is assumed to be a known function of V, the bubble volume. We assume that this function is the adiabatic one, $\pi = KV^{-\delta}$, where K and δ are constants, δ being the adiabatic exponent for the gas within the bubble. Then because pressure must be continuous across the bubble boundary, we have the dynamic condition $p = KV^{-\delta}$ or, using (3)

(10) $p_o - \rho\left[\phi_t + \frac{1}{2}(\nabla\phi)^2\right] - \rho g z = KV^{-\delta}$ on $F = 0$.

The mathematical problem which we consider is that of finding ϕ and F, satisfying (2,5,9,10), given $F(x,y,z,o)$ and $F_t(x,y,z,o)$. It is not necessary to specify ϕ initially since it is determined by (2,5,9). In particular, we will assume that the bubble is initially a sphere of raduis A_o and that its initial radial velocity is a constant $\dot{A}_o$. Once ϕ is determined, $\vec{u}$ and p are given by (1,3).

III Coordinates

Suppose that the origin of the x,y,z coordinate system is fixed at the center of the bubble at t = 0. We now introduce a moving coordinate system ξ,η,ζ by the equations

(11) $\xi = x, \eta = y, \zeta = z - B(t)$.

In (11) B(t) is a function to be determined subsequently in such a way that the origin is always at the center of gravity of the bubble. Moving coordinates are introduced in order that the origin remain inside the bubble as long as possible.

Because of the cylindrical symmetry of the problem it is advantageous to introduce the spherical coordinates r,θ,ω. Obviously the solution will be independent of ω, and we will assume this from the outset. We define r,θ,ω by

(12) $$r^2 = \xi^2 + \eta^2 + \zeta^2 = x^2 + y^2 + (z-B)^2 \quad , \quad r \geq 0$$

$$\cos\theta = \frac{\xi}{r} = \frac{z-B}{\sqrt{x^2 + y^2 + (z-B)^2}} \quad , \quad 0 \leq \theta \leq \pi$$

$$\tan\omega = \frac{\xi}{\eta} = \frac{x}{y} \quad , \quad 0 < \omega < 2\pi$$

Now introducing $R(\theta,t)$ we write the equation of the bubble surface in the form

(13) $$F(x,y,z,t) = r - R(\theta,t) = 0$$

The bubble volume is then given by

(14) $$V(t) = \frac{2\pi}{3} \int_0^\pi R^3(\theta,t)\sin\theta d\theta$$

The condition that the moving origin be at the center of gravity of the bubble may be written as

(15) $$\int_0^\pi R^4 \cos\theta d\theta = 0$$

To express (2,9,10) in terms of r and θ, we observe that if $\bar{I}(\xi,\eta,\zeta,t)$ is the expression in terms of ξ,η,ζ for some function $I(x,y,z,t)$, i.e. if $I(x,y,z,t) \equiv \bar{I}(\xi,\eta,\zeta,t)$, then we have the following differentiation formulae:

(16) $$I_x = \bar{I}_\xi, \; I_y = \bar{I}_\eta, \; I_z = \bar{I}_\zeta, \; I_t = I_t - \dot{B}\bar{I}_\zeta = \bar{I}_t - \dot{B}(\cos\theta \bar{I}_r - \frac{\sin\theta}{r}\bar{I}_\theta)$$

If we apply (16) to (9,10) we obtain

(17) $$\phi_r - \frac{\theta}{r^2}\phi_\theta = R_t + \dot{B}(\cos\theta + \frac{\sin\theta}{r}R_\theta) \quad \text{at } r = R(\theta,t)$$

(18) $$K\left(\frac{2\pi}{3}\int_0^\pi R^3 \sin\theta d\theta\right)^{-\delta} = P_0 - \rho\Big[g(r\cos\theta + B) + \phi_t - \dot{B}(\cos\theta\phi_r - \frac{\sin\theta}{r}\phi_\theta) + \frac{1}{2}(\phi_r^2 + \frac{1}{r^2}\phi_\theta)\Big] \quad \text{at } r = R(\theta,t).$$

The problem is now that of finding $\phi, R(\theta,t)$ and $B(t)$ satisfying (2,5,15,17,18) and the initial conditions (19)

$$(19) \quad R(\theta,o)=A_o, \quad R_t(\theta,o)=\dot{A}_o, \quad B(o)=\dot{B}(o)=0$$

It will prove convenient to employ the energy equation, which is a direct consequence of the preceding equations, namely

$$(20) \; \frac{K}{\delta-1}\left(\frac{2\pi}{3}\int_o^{\pi} R^3\sin\theta d\theta\right)^{1-\delta} + \frac{2\pi}{3}(P_o-\rho gB)\int_o^{\pi} R^3\sin\theta d\theta -$$

$$-\pi\rho\int_o^{\pi} R^2(R_t+\dot{B}[\cos\theta+\frac{\sin\theta}{r}R_\theta])\phi(R,\theta,t)\sin\theta d\theta =$$

$$= \frac{4\pi}{3}A_o^3\left[\frac{K}{\delta-1}\left(\frac{4\pi}{3}A_o^3\right)^{-\delta}+P_o+\frac{3}{2}\rho\dot{A}_o^2\right] = E$$

The terms on the left side are the internal energy of the bubble, the potential energy and the kinetic energy of the water, respectively: the terms on the right are the corresponding quantities evaluated at $t = 0$, and E is the total energy which is defined by this equation.

IV Dimensionless Variables

It is convenient to define dimensionless variables $\bar{r},\bar{t},\bar{\phi},\lambda,b,\sigma,a_o,\dot{a}_o,k$ by introducing a unit of length L and a unit of time T. These quantities are defined as follows:

$$(21) \quad r = L\bar{r} \qquad B(t) = Lb(\bar{t}) \qquad k = \frac{K\beta_o^{\delta-}E^{-\delta}}{\delta-1}$$

$$t = T\bar{t} \qquad\qquad = L\rho gP_o^{-1}$$

$$\phi(r,\theta,t) = L^2T^{-1}\bar{\phi}(\bar{r},\theta,\bar{t}) \qquad A_o = La_o \qquad \hat{k} = \frac{1}{\delta-1}\left(\frac{\delta-1}{\delta}\right)^{\delta}$$

$$R(\theta,t) = L\lambda(\theta,\bar{t}) \qquad \dot{A}_o = LT^{-1}\dot{a}_o$$

For L and T we choose the definitions

$$(22)\ L = \left[\frac{3(\delta-1)E}{4\pi\delta P_o}\right]^{1/3}, \qquad T = L\sqrt{\frac{\rho}{P_o}}$$

Now equations(2,5,15,17,18,19) become the following set for ϕ, λ and b:

$$(23)\quad \nabla^2\phi = 0\ , \quad \phi \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$(24)\ g(\theta,t) \equiv \phi_r - \frac{\lambda_\theta}{r^2}\phi_\theta - \lambda_t - \dot{b}(\cos\theta + \frac{\sin\theta}{r}\lambda_\theta) = 0$$

at $r = \lambda(\theta,t)$

$$(25)\ h(\theta,t) \equiv \phi_t + \frac{1}{2}\Big([1 + (\frac{\lambda_\theta}{\lambda})^2][\frac{1}{r}\phi_\theta + \dot{b}\sin\theta]^2 +$$

$$+2\lambda_t\frac{\lambda_\theta}{\lambda}[\frac{1}{r}\phi_\theta + \dot{b}\sin\theta] + \lambda_t^2 - \dot{b}^2\Big) + \sigma(\lambda P_1 + b) +$$

$$+ \frac{k}{\hat{k}}\left(\frac{1}{2}\int_0^\pi \lambda^3\sin\theta d\theta\right)^{-\delta} - 1 = 0 \quad \text{at } r = \lambda(\theta,t).$$

$$(26)\quad \lambda(\theta,o) = a_o,\ \lambda_t(\theta,o) = \dot{a}_o\ ,\ b(o) = \dot{b}(o) = 0$$

$$(27)\quad \int_0^\pi \lambda^4\cos\theta\ \sin\theta d\theta = 0$$

In (23-27) the bars have been omitted and two new functions $g(\theta,t)$ and $h(\theta,t)$ have been defined by (24,25). The quantity $\hat{k}$ is defined in (21). Equation (25) is a consequence of both (7) and (8), and the symbol P, in it denotes the Legendre polynomial $P_1(\cos\theta) = \cos\theta$.

The energy equation (20) becomes, in dimensionless variables and omitting the bars,

$$(28) - \frac{3}{4}\int_0^\pi \lambda^2\left(\lambda_t + \dot{b}[\cos\theta + \frac{\sin\theta}{\lambda}\lambda_\theta]\right)\phi(\lambda,\theta,t)\sin\theta\ d\theta +$$

$$+ \frac{1}{2}(1 - \sigma b)\int_0^\pi \lambda^3\sin\theta\ d\theta \ + \hat{\alpha}^{-3\delta}k(\frac{1}{2}\int_0^\pi \lambda^3\sin\theta\ d\theta)^{1-\delta} = \hat{\alpha}^{-3}$$

It should be noted that a_o and $\dot{a}_o$ are no longer independent, since they are related by the following equation, obtained by setting t = o in (28).

$$(28') \quad \frac{3}{2} a_o^3 \dot{a}_o^2 + a_o^3 + k \hat{\alpha}^{-3\delta} a_o^{3(1-\delta)} = \hat{\alpha}^{-3}$$

Here $\hat{\alpha}$ is the constant

$$(29) \quad \hat{\alpha} = \frac{\delta-1}{\delta} 1/2$$

V Method of Solution

In order to solve (23-27) for $\phi(r,\theta,t)$, $\lambda(\theta,t)$, and $b(t)$, we make use of the fact that a solution ϕ of (23) can be expressed as a linear superposition of zonal harmonics $r^{-(m+1)}P_m(\cos\theta)$ and that λ can be expressed as a linear superposition of Legendre polynomials $P_m(\cos\theta)$. The coefficients in these expansions are functions of t. Next we make the assumption that all these coefficients, as well as $b(t)$, can be expressed as series(convergent of asymptotic) of powers of the parameter σ. This parameter, defined in(21), is proportional to the acceleration of gravity g, and is equal to the ratio of the unit of length L to the hydrostatic head $P_o/\rho g$ at the initial bubble center. Thus when $\sigma = 0$ the present solution is just that of the spherical bubble theory since $\sigma = 0$ is equivalent to $g = 0$. Formally we assume

$$(30) \quad \phi(r,\theta,t) = \sum_{n=o}^{\infty} \sigma^n \sum_{m=o}^{\infty} c_{nm}(t)\, r^{-(m+1)} P_m(\cos\theta)$$

$$(31) \quad \lambda(\theta,t) = \sum_{n=o}^{\infty} \sigma^n \sum_{m=o}^{\infty} a_{nm}(t)\, P_m(\cos\theta)$$

$$(32) \quad b(t) = \sum_{n=o}^{\infty} \sigma^n b_n(t)$$

The problem is now reduced to that of determining c_{nm}, a_{nm} and b_n which are all functions of t, in such a way that (30-32) satisfy (24-27), since (23) is already satisfied. Insertion of (31,32) into (26) yields (dot denoting t derivative)

$$(33)\quad a_{oo}(o) = a_o, \quad \dot{a}_{oo}(o) = \dot{a}_o$$

$$a_{nm}(o) = o, \quad \dot{a}_{nm}(o) = o \quad \text{if} \quad n + m > o$$

$$(34)\quad b_n(o) = 0, \quad \dot{b}_n(o) = 0, \quad n \geq 0$$

We now insert (30-32) into (24,25) and equate coefficients of $\sigma^n P_m(\cos\theta)$, thus obtaining a sequence of equations involving the coefficients and their first time derivatives.

The zero order terms ($n = o$) correspond to the solution obtained when $\sigma = o$, i.e. in the absence of gravity. The equations for these coefficients, with the corresponding initial conditions from (33) can be solved at once and yield

$$(35)\quad c_{om} \equiv o, \quad a_{om} \equiv o, \quad \text{for} \quad m \geq 1$$

$$b_o \equiv o$$

The other two zero order coefficients satisfy

$$(36)\quad c_{oo} = -(a_{oo}^2\, \dot{a}_{oo})$$

$$(37)\quad (a_{oo}^2\, \dot{a}_{oo})^{\cdot} a_{oo}^{-1} - \tfrac{1}{2}\dot{a}_{oo}^2 - \frac{k}{\bar{k}}\, a_{oo}^{-3\delta} = 1$$

Equation (37) is the well known equation of the classical theory of spherical bubbles, expressed in terms of our units, and (36) is the corresponding expression for the potential. The solution of (37) satisfying the initial conditions (33) is a periodic function of t which increases monotonically from a minimum value to a maximum value during a half period and decreases monotonically to the minimum during the next half period. The solution can be written in the form

$$(38)\qquad a_{oo}(t) = \hat{\alpha}^{-1}\alpha\left(\sqrt{\tfrac{2}{3}}\,\hat{\alpha}t + \tau_o\right) \equiv a(t),$$

where the phase t_o is defined by $\hat{\alpha}^{-1}\alpha(\tau_o) = a_o$

The function $\alpha(\tau)$ is an even periodic function of τ defined over the first half period by its inverse function $\tau(\alpha)$

$$(39)\qquad \tau(a) = \int_{\underline{\alpha}}^{a} \frac{dx}{\sqrt{x^{-3}-1-kx^{-3\delta}}} \qquad \underline{\alpha} \le a \le \bar{\alpha}$$

The numbers $\underline{\alpha}$ and $\bar{\alpha}$, $\underline{\alpha} \le \bar{\alpha}$, are the minimum and maximum values of α, and are the two roots of the equation

$$(40)\qquad 1 - x^3 - kx^{-3(\delta-1)} = 0$$

The parameter k, which determines $\underline{\alpha}$ and $\bar{\alpha}$, is defined in (21) and satisfies the inequality

$$(41)\qquad 0 < k \le \hat{k} = \frac{1}{\delta-1}\left(\frac{\delta-1}{\delta}\right)^{\delta}$$

The equality occurs only when $\dot{a}_o = 0$ and $P_o = k\left(\frac{4\pi}{3}A_o^3\right)^{-\delta}$. The second condition means that the initial bubble pressure equals the initial hydrostatic pressure at its center. From (40) we also have the inequality [$\hat{\alpha}$ is defined in (29)].

$$(42)\qquad 0 < \underline{\alpha}(k) \le \hat{\alpha} \le \bar{\alpha}(k) < 1$$

The equalities holds when $k = \hat{k}$ in which case we have

$$(43)\qquad \alpha(\tau) \equiv \hat{\alpha} = \text{constant}$$

Then from (38) $a_{oo}(t)$, which we will henceforth call $a(t)$, is constant and equal to one. Thus the unit of length L is the equilibrium bubble radius, i.e. the radius at which the bubble pressure would equal the hydrostatic pressure corresponding to its initial center, while the total energy E would be unchanged.

In order to write the equations for the higher order coefficients ($n \geq 1$) it is convenient to introduce the functions $\bar{g}(\theta,t)$ and $\bar{h}(\theta,t)$ defined by

$$(44)\quad \bar{g}(\theta,t) \equiv g(\theta,t) + \sum_{n=1}^{\infty} \sigma^{n}\Big(\dot{b}_n P_1 + \sum_{m=o}^{\infty} \big[\dot{a}_{nm} + 2\frac{\dot{a}}{a} a_{nm} + (m+1)\, a^{-(m+2)} \subset_{nm}\big] P_m\Big) = \sum_{n=o}^{\infty} \sigma^{n} \sum_{m=o}^{\infty} g_{nm} P_m$$

$$(45)\quad \bar{h}(\theta,t) \equiv h(\theta,t) - \sum_{n=1}^{\infty} \sigma^{n}\Big(-3\delta \frac{k}{\hat{k}} a^{-3\delta-1} a_{no} + \sum_{m=o}^{\infty} \big[\frac{(a^2\dot{a})^{\cdot}}{a^2} a_{nm} + \dot{a}\,\dot{a}_{nm} + a^{-(m+1)} \dot{\subset}_{nm}\big] P_m\Big) = \sum_{n=o}^{\infty} \sigma^{n} \sum_{m=o}^{\infty} h_{nm} P_m$$

The functions $g(\theta,t)$, $h(\theta,t)$ occurring in (44,45) are defined in (24,25). The coefficients g_{nm}, h_{nm} are defined by the above equations. It is easily verified that g_{nm} and h_{nm} depend only upon these coefficients a_{sm}, $\subset_{sm}$, b_s with $s < n$.

Now inserting (30-32) into (24) we obtain for $n > 0$

$$\subset_{nm} = -\frac{1}{m+1} a^{m+2} \big[\dot{a}_{nm} + 2\frac{\dot{a}}{a} a_{nm} - g_{nm}\big] \quad , m \neq 1$$

(46)

$$\subset_{n1} = -\frac{1}{2} a^{3}\big[\dot{b}_n + \dot{a}_{n1} + 2\frac{\dot{a}}{a} a_{n1} - g_{n1}\big] \quad , m = 1$$

Similarly inserting (30-32) into (25) and making use of

(45,46) we finally obtain after some manipulation

$$a\ddot{a}_{no} + 3\dot{a}\dot{a}_{no} + (\ddot{a} + 3\delta \frac{k}{\hat{k}} a^{-3\delta-1})a_{no} = a\dot{g}_{no} + 2\dot{a}g_{no} + h_{no} \,, \qquad m = 0$$

(47)

$$(a^3\dot{b}_n)^{\cdot} = (a^3[g_{n1} - \dot{a}_{n1}]) + 2a^2 h_{n1} \,, \qquad m = 1$$

$$a\ddot{a}_{nm} + 3\dot{a}\dot{a}_{nm} + (1-m)\ddot{a}a_{nm} = a\dot{g}_{nm} + (m+2)\dot{a}g_{nm} + (m+1)h_{nm} \,, \qquad m>1$$

Equations (47) and the initial conditions (33,34) determine the $a_{nm}(m\neq 1)$ and b_n once a_{km} and $b_k(k<n)$ are known. The a_{n1} are determined in terms of the a_{km} with $k<n$ by (27). The c_{nm} are given by (46).

Now all the $a_{om}(m\geq 1)$ are zero, and $b_o=0$, by (35), and $a_{oo} \equiv a(t)$ is given by (38,39). Thus we may proceed to find the a_{1m} and b_1 and continue step by step to obtain all the coefficients. It would seem from (47) that at each step $n>o$ we have an infinite set of non-homogeneous ordinary differential equations to solve. However it is shown in Appendix I that a_{n1}, g_{nm} and h_{nm} are zero whenever $m>n$ or $m+n$ = odd integer. Thus those of equations (47) in which these quantities appear become homogeneous linear equations, and since the initial data are zero by (33,34) the solutions of these equations are zero. Therefore we have

(48) $a_{nm} \equiv 0$ if $m>n$ or $n+m$ = odd integer

$b_n \equiv 0$ if n = even integer

Thus from (46)

(49) $c_{nm} \equiv 0$ if $m>n$ or $n+m$ = odd integer

Hence at each step there is only a finite set of equations (exactly $[\frac{n+1}{2}]$) to be solved.

As a consequence the expansions for ϕ, λ, and b, up to the third power of σ, simplify to

$$(50)\quad \phi = -a^2\dot{a}r^{-1} + \sigma(c_{11}r^{-2}P_1) + \sigma^2(c_{20}r^{-1} + c_{22}r^{-3}P_2)$$

$$+ \sigma^3(c_{31}r^{-2}P_1 + {}_{33}r^{-4}P_3) + \cdots$$

$$\lambda = a + \sigma^2(a_{20} + a_{22}P_2) + \sigma^3(a_{33}P_3) + \cdots$$

$$b = \sigma b_1 + \sigma^3 b_3 + \cdots$$

In the expansion for λ the terms $\sigma a_{11}P_1$ and $\sigma^3 a_{31}P_1$ were omitted, since by (27) $a_{11} \equiv a_{31} \equiv 0$. It is not true, however, that $a_{n1} \equiv 0$ for all n. In fact, for n = 5 we have $a_{51} = -\frac{27}{70} a^{-1} a_{22} a_{33}$.

It is convenient to take advantage of the preceding results, at least up to terms in σ^2, in order to calculate $\bar{g}(\theta,t)$ and $\bar{h}(\theta,t)$, which then become

$$(51)\quad \bar{g}(\theta,t) \equiv -3a^{-1}\dot{b}_1(a_{20} - \tfrac{1}{5}a_{22})P_1\sigma^3 - \tfrac{18}{5}a^{-1}\dot{b}_1 a_{22}P_3{}^3 + \cdots$$

$$\bar{h}(\theta,t) \equiv aP_1\sigma + (b_1 + \tfrac{1}{4}\dot{b}_1^2)\sigma^2 - \tfrac{3}{4}\dot{b}_1^2 P_2{}^2 +$$

$$\tfrac{3}{5}(5a_{20} + 2a_{22} + a\dot{b}_1[a^{-1}a_{22}]^{\cdot})P_1\sigma^3 +$$

$$+ \tfrac{3}{5}(3a_{22} - a\dot{b}_1[a^{-1}a_{22}]^{\cdot})P_3\sigma^3 + \cdots$$

These expressions facilitate writing equations (47) explicitly.

VI <u>Solution Up to σ^3</u>

For $n = 1$ the only non-zero coefficient to be determined is b_1. Since from (51) we have $c_{11} = o$, $h_{11} = a$, equation (47) yields for b_1

$$(52) \quad (a^3 \dot{b}_1)^{\bullet} = 2a^3$$

Thus, using the initial data in (34), we have

$$(53) \quad b_1 = 2 \int_0^t a^{-3} \left[\int_0^t a^3 \, d\tau \right] d\tau$$

Equation (53) is the Herring rise formula which accounts for the only first order effect. The bubble shape is unaffected in first order, but the water flow is modified since

$$c_{11} = - \int^t a^3 \, dt.$$

For $n = 2$ the only coefficients to be determined are a_{20} and a_{22}. From (47,51,53) we have the equations

$$(54) \quad a\ddot{a}_{22} + 3\dot{a}\dot{a}_{20} + (\ddot{a} + 3\delta \frac{k}{\hat{k}} a^{-3\delta-1}) a_{20} = b_1 + \frac{1}{4} \dot{b}_1^2 ,$$

$$a_{20}^{(o)} = \dot{a}_{20}^{(o)} = 0$$

$$(55) \quad a\ddot{a}_{22} + 3\dot{a}\dot{a}_{22} - \ddot{a}a_{22} = -\frac{9}{4} \dot{b}_1^2 , \quad a_{22}^{(o)} = \dot{a}_{22}^{(o)} = 0$$

In terms of the solutions of these equations and making use of (46), the only non-zero coefficients c_{2m} are given by

$$(56) \quad c_{20} = - (a^2 a_{20})^{\bullet}$$

$$(57) \quad c_{22} = - \frac{1}{3} a^2 (a^2 a_{22})^{\bullet}$$

For $n = 3$ the only coefficients to be determined are a_{33} and b_3. From (47,51,33) we have

(58) $a\ddot{a}_{33} + 3\dot{a}\dot{a}_{33} - 2\ddot{a}a_{33} = -6\dot{b}_1(\dot{a}_{22} + \frac{1}{5}\dot{a}a^{-1}a_{22})$,

$$a_{33}(0) = \dot{a}_{33}(0) = 0$$

(59) $(a^3\dot{b}_3)^{\cdot} = -3(a^2\dot{b}_1 a_{20})^{\cdot} + 6a^2 a_{20} + \frac{9}{5}(a^2\dot{b}_1 a_{22})^{\cdot}$,

$$b_3(0) = \dot{b}_3(0) = 0$$

Then using (48)

(60) $c_{31} = -\frac{1}{2}[a^3\dot{b}_3 + 3\dot{b}_1 a^2(a_{20} - \frac{1}{5}a_{22})]$

(61) $c_{33} = -\frac{1}{4}[a^3(a^2 a_{23})^{\cdot} + \frac{18}{5} b_1 a^4 a_{22}]$

The preceding equations can be solved explicitly in the special case of the so-called "equilibrium bubble", which is treated in the next section. In general, however, it seems necessary to integrate the preceding equations numerically in order to obtain quantitative results. Such results will be published soon.*

It is useful to recognize that equation (53) for the rise $b_1(t)$ can be put in the form

(62) $b_1(t) = S\,t^2 + P\,t + Q$

Here S is a constant while P and Q are periodic functions of t having the same period as a(t). This form permits us to determine the behavior of $b_1(t)$ for all t by merely evaluating integrals for one period. These integrals appear in the definitions of S, P and Q which are, if t_0 is the period of a(t)

*Penny and Price have examined the solution of the homogeneous equations (47) for n = 1 with non-zero initial data in order to study the growth of initial asymmetry.

$$(63) \qquad S = \left[\frac{1}{t_o}\int_o^{t_o} a^3 d\tau\right]\left[\frac{1}{t_o}\int_o^{t_o} a^{-3} d\tau\right]$$

$$P = 2\left[\frac{1}{t_o}\int_o^{t_o} a^3 d\tau\right]\left(\int_o^{t} a^{-3} d\tau - \left[\frac{1}{t_o}\int_o^{t_o} a^{-3} d\tau\right] t\right)$$

$$Q = b_1(t) - St^2 - Pt$$

The periodicity of P is obvious, while that of can be proved with some manipulation. Equations (63) apply only if $\tau_o = \dot{\lambda}_o = \dot{a}_o = 0$.

VII The Equilibrium Bubble

In section V, following equation (41) it is shown that if $k = \hat{k}$ then the zero order bubble radius $a(t)$ remains constant and equal to one. This occurs if the initial radial velocity is zero and if the initial bubble pressure equals the pressure in the water at the depth of the bubble center. We refer to this case as that of the equilibrium bubble since the bubble would remain at rest in the absence of gravity.

Since the radius $a(t)$ is identically constant and in fact equal to one, due to our choice of units, all the equations of the preceding section simplify to such an extent that they can be solved explicitly. By solving them we obtain the following expressions for $\lambda(\theta,t)$, $b(t)$ and $\phi(r,\theta,t)$ up to and including the third order in

$$(64) \qquad \lambda(\theta,t) = 1 + (\sigma t^2)^2 \left[f(\tau) - 3/4\, P_2(\cos\theta)\right] + (\sigma t^2)^3 \frac{6}{5} P_3(\cos\theta) + \ldots$$

$$(65) \qquad b(t) = \sigma t^2 - (\sigma t^2)^3 g(\tau) + \ldots$$

$$(66) \quad \phi(r,\theta,t) = -\sigma t r^{-2} P_1(\cos\theta) - \frac{4}{3\delta}\left(t - \frac{\sin\sqrt{3\delta}\,t}{\sqrt{3\delta}}\right)\sigma^2 r^{-3}$$

$$+ \sigma^2 t^3 r^{-3} P_2(\cos\theta) + \left(\frac{9}{10}t^5 - \frac{2}{3\delta}t^3 + \frac{12}{9\delta^2}t\right.$$

$$\left. - \frac{12}{(3\delta)^{5/2}} \sin\sqrt{3\delta}\,t\right)\sigma^3 r^{-4} P_1(\cos\theta) \qquad 3$$

$$- \frac{9}{20}\sigma^3 t^5 r^{-4} P_3(\cos\theta) \quad ..$$

In equations (64,65) $\tau = \sqrt{3\delta}\,t$ and $f(\tau)$, $g(\tau)$ are given by

$$(67) \quad f(\tau) = \frac{2}{\tau^2} - \frac{4}{\tau^4}(1 - \cos\tau)$$

$$(68) \quad g(\tau) = \frac{9}{20} - \frac{2}{\tau^2} - \frac{24}{\tau^6}\left[2(1 - \cos\tau) - \tau\sin\tau\right]$$

These functions both decrease fairly slowly from $f(o) = 1/6$, $g(o) = 7/12$ to $f(\infty) = 0$, $g(\infty) = 9/20$.

Using equations (64) and (65) we have constructed graphs of the bubble profile at various times from $t = 0$ until $t = 1.09$, choosing $\sigma = .222$. These graphs, in Figure I , show the rise of the bubble as well as its deformation.

Since (64,65) are only the first terms in a series solution, we can expect them to represent the solution accurately for a short time, after which the omitted terms become important. In order to determine when the omitted terms become significant, we can examine the expressions in (64,65) to find when they fail. Two kinds of failure occur. In the first kind, the origin of coordinates, which is located at the center of gravity of the bubble, may cross the bubble surface and enter the water. Then the potential function $\phi(r,\theta,t)$ which has singularities at the origin, will be singular in the water, contrary to the assumption of regularity on which the solution was based. Therefore the expressions (64,65) certainly become invalid as soon as $\lambda = 0$ for some values of θ and t. If $\sigma \geq .222$ this failure is the first to occur, and it occurs when $\sigma t^2 \simeq .8$ and for $\theta = \pi$. Thus in this case the bottom of the bubble bulges upward until it reaches the center of gravity.

The second type of failure occurs because (65) predicts that the bubble will rise for awhile and then fall. This is, of course, physically unreasonable, and we therefore consider (65) to fail when $\dot{b}(t) = 0$, i.e. when the bubble stops rising. This type of failure is the first to occur if $\sigma \leq .222$, and it occurs when $\sigma t^2 \simeq .8$. Therefore when $\sigma = .222$ both failures occur simultaneously, and this occurs at t = 1.89, which explains why the graphs in Figure I were only constructed until this value of t.

As an example of these results, let us consider a bubble of one inch radius initially located ten inches below the surface.

$$(A_o = 1'' \ , \ Z_o = 10'')$$

The pressure above the surface is atmospheric (p_o = 1 atmosphere) so that the hydrostatic head at the bubble is 33 feet plus 10 inches or 406 inches, which is 406 times the initial radius. Thus $\sigma = \frac{1}{406} = .00246$. The unit of length L is equal to the initial radius A_o for the equilibrium bubble, so $L = A_o = 1$ inch. Then the unit of time T = .0025 seconds. If we use the Herring rise formula, which is the first term of (65), the rise $b(t) = \sigma t^2$, so the bubble will reach the surface when $\sigma t^2 = \frac{Z_o}{L} = 10$ or t = 64 units. However equation (64) fails when $\lambda(\pi,t) = 0$ which occurs at t = 16 units or .040 seconds. The bubble rises only .55 inches before the formulae fail.

VIII Discussion of Results

We have presented a systematic method for determining the shape and rise of the bubble and the potential function in the water, from which the pressure can be computed. This method is based upon a power series expansion in terms of a dimensionless parameter σ which is the ratio of the equilibrium bubble radius to the hydrostatic head above the initial bubble center. Thus σ is proportional to the acceleration of gravity g, so our expansion can also be considered to be in powers of g. For any bubble which does not vent (i.e. break through the water-air surface) during its first expansion, σ is less than one and greater than zero.

The terms of zero order in σ correspond to the classical theory of a spherical bubble. They describe a solution in which the bubble remains spherical and performs undamped periodic radial oscillations, while its center remains at rest. The equilibrium bubble is a special case which does not even oscillate. When terms of first order are also included, the only modification is that the center of the bubble rises according to the Herring rise formula.

When terms of second order are included the shape of the bubble is found to change although the rise formula is unmodified. The changes in the bubble are of two kinds corresponding to the two terms

$\sigma^2 a_{20}(t)$ and $\sigma^2 a_{22}(t) P_2(\cos\theta)$ in (50) respectively. The first term corresponds to a change in volume of the bubble, which is an increase in the case of the equilibrium bubble and also in a numerical example which we have treated. Such an increase is to be expected in general on physical grounds, since the bubble is rising into a region of lower pressure. From the differential equation(54) for $a_{20}(t)$ it can be shown that in general a_{20} starts at zero and increases, at least initially, in agreement with the preceding considerations. As a consequence of the increase of $a_{20}(t)$ it follows that the time between successive minima (or maxima) of the bubble volume is decreased below that given by the zero order (spherical bubble) term alone. In addition the expansion phase is increased in duration, while the contraction phase is diminished in duration.

The other second order term corresponds to a flattening of the bubble. Since $a_{22}(t)$ starts at zero and decreases, as can be seen from the differential equation(55), this flattening is in the vertical direction, i.e. the vertical separation between top and bottom is diminished, while the horizontal separation between sides is increased. This can be seen in the figures, and is in agreement with observation, as are the preceding results.

When terms of third order in σ are included the shape of the bubble is again modified while the rise formula is also changed. The only term of this order in the equation for the bubble radius is $\sigma^3\ a_{33}(t)\ P_3(\cos\theta)$, which leaves the bubble volume unchanged but changes its shape. Since $a_{33}(t)$ starts at zero and increases, as one can show from the differential equation (58), this change of shape corresponds to pulling up the bottom and pushing up the top, so the bubble becomes kidney-shaped with the lower side concave. This behavior can be seen in the figures and is also in agreement with observation. The third order correction to the rise formula can be shown to start at zero and to become negative, from equation (59), in agreement with the numerical example and the equilibrium bubble. This correction is also in agreement with experiment, since the Herring formula predicts too large a rise.

All of the gravity effects described here are greater in magnitude and occur more quickly, the larger the value of σ. Now σ is larger for large explosions, so all the effects described should be more prominent for such explosions. A large value of σ is also obtained if the pressure above the water surface is reduced. This explains why these effects have been observed in reduced pressure tanks.

The behavior of the bubble depends not only upon σ but also upon another dimensionless parameter k. The smaller the value of k the greater is the amplitude of oscillation of the bubble, since small k corresponds to a large explosion energy

$$\left(k = \frac{K\, P_o^{\delta-1}\, E^{-\delta}}{\delta-1}\right).$$

The maximum value of k is $\hat{k}$ which is attained for the equilibrium bubble. Since the gravitational effects depend mainly on σ, results for the equilibrium bubble indicate what will happen in general. Therefore we expect that for any value of k, equations (50) for λ and b will fail at a certain time, just as they do for the equilibrium bubble, $k = \hat{k}$. Since in general the bubble becomes flattened and then kidney-shaped, we expect the center part of the lower surface to continue rising, like a jet, until it reaches the upper surface, thus converting the bubble into a torus. For the equilibrium bubble, failure of the equations occurs when $\sigma t^2 \simeq .8$, so we may assume that the real breakdown for any bubble will occur when $\sigma t^2 \simeq 1$ or $t \simeq \sigma^{-1/2}$. As one period of the bubble is about 3/2 units for an explosion bubble, the breakup should occur after about $\frac{2}{3}\sigma^{-1/2}$ periods. For large explosions, with large values of σ, this may occur during the first period.

If the bubble does breakdown into a torus, what will be its subsequent behavior? The results of

Tomotika on a torus of one fluid within another fluid indicate that a torus is also unstable and will generally breakup into a certain number of pieces determined by the dimensions, etc. Then each of these pieces might flatten, become kidney shaped and breakdown into another torus, which would again breakup, etc. This seems to happen when a drop of one fluid (e.g. ink) is dropped into another (e.g. water).

Finally let us consider how compressibility will alter the present results. We have attempted to account for compressibility of the water, as we did in our previous report on the spherical bubble, by using the wave equation for the potential in the water rather than Laplace's equation.* We then expressed the solution of this equation as an infinite series of multipoles, but we were forced to use a fixed origin rather than a moving one. We also expressed the bubble surface as a series in powers of σ and in Legendre polynomials, and attempted to determine the coefficients. We found that the terms of zero order in σ were exactly the same as the solution of our previous report, as we naturally expected. The higher order coefficients satisfied second order equations similar to those obtained in the present report, but the coefficients in these equations involved the function a(t) of our previous report, rather than the incompressible solution a(t). This was not the only difference but seemed to be the most important one. We therefore expect that all of the present results will still apply in a compressible fluid, and that the main effects of compressibility would be accounted for by replacing a(t) in all equations by the compressible solution of our previous report. In this way the main result of that report-damped radial oscillation-would be incorporated into the present work.

*In an unpublished manuscript.

Appendix I

Theorem: If ϕ, λ and b are represented by the formal series in equations (30-32) and if they satisfy the equations (24-27) then for all $n \geq 0$

(A1) $g_{nm} = h_{nm} = a_{nm} = c_{nm} = 0$

if $m > n$ or if $m + n$ is odd, and

(A2) $b_n = 0$

if n is zero or even.

Furthermore g_{nm} and h_{nm} are polynomials in $a_{k\ell}$, $\dot{a}_{k\ell}$, b_k and $\dot{b}_k$ with $0 < \ell \leq k < n$ and are independent of all other a's and b's except a_{oo}.

Proof: The proof relies on induction with respect to n. Thus we assume that the theorem is true for all $n \leq N$ and we will show that this implies the theorem for $n = N + 1$. It has already been shown in the text that the theorem is true for $n = 0$ (see equations (35) and (51)) and therefore the theorem will follow for all n.

In order to perform the inductive step, we introduce the set H_k^ℓ of all formal power series in σ and x with coefficients depending upon a parameter t, which satisfy the conditions $u_{nm}(t) = 0$ if $n < k$, or $m > n$ and $n \leq \ell$, or $m + n$ = odd and $n \leq \ell$. The quantities $u_{nm}(t)$ are the coefficients in such a formal series u given by

(A3) $$u = \sum_{n=0}^{\infty} \sigma^n \sum_{m=0}^{\infty} x^m u_{nm}(t).$$

We also define the subset $\bar{H}_k^\ell \subset H_k^\ell$ of power series in which the non-zero coefficients u_{nm}, $n \leq \ell$, are polynomials over some set [c].

If $x = \cos\theta$, then since $P_n(x)$ is even or odd according as n is even or odd it follows from the induction hypothesis that

(A4) $\bar{g} \in H_o^N, \quad \bar{h} \in H_o^N.$

If we let [c] denote the set of a_{nm}, c_{nm}, b_n and their time derivatives with $0 < n \leq N$ then it also follows from the induction hypothesis that

(A5) $\phi(r,t) \in \bar{H}_o^N, \quad \lambda \in \bar{H}_o^N, \quad bx \in \bar{H}_1^N.$

We first wish to show that

(A6) $\bar{g} \in \bar{H}_o^{N+1}, \quad \bar{h} \in \bar{H}_o^{N+1}.$

In order to do this we first note the following properties of the sets $\bar{H}_k^{\ell}$:

(a) $\bar{H}_k^{\ell} \subset \bar{H}_{k'}^{\ell'}$ if $\ell \geq \ell'$ and $k \geq k'$

(b) $u \in \bar{H}_o^{\ell} \rightarrow u_{nm}\sigma^n x^m \in \bar{H}_n^{\infty} \quad n \leq \ell$

(c) $u \in \bar{H}_k^{\ell}, \quad v \in \bar{H}_k^{\ell} \rightarrow (\alpha u + \beta v) \in \bar{H}_k^{\ell}, \quad uv \in \bar{H}_{2k}^{\ell+k}$

(d) $u \in \bar{H}_k^{\ell}, \quad v \in \bar{H}_{k'}^{\ell'}, \quad \ell + k' \leq \ell' + k \rightarrow uv \in \bar{H}_{k+k'}^{\ell+k'}$

(e) $u \in \bar{H}_1^{\ell}, \quad \alpha \neq 0 \rightarrow (\alpha + u)^n - n\alpha^{n-1}u \in \bar{H}_o^{\ell+1},$

$(\alpha + u)^n - \alpha^n \in \bar{H}_1^{\ell}$

(f) $u \in \bar{H}_o^{\ell} \rightarrow u_t \in \bar{H}_o^{\ell}$

(g) $u \in \bar{H}_o^{\ell}, \quad v \in \bar{H}_o^{\ell} \rightarrow (1 - x^2)u_x v_x \in \bar{H}_2^{\ell+1}.$

Now making use of these properties and the induction hypothesis we can prove (A6).

To prove (A6) we consider various parts of $\bar{g}$ and $\bar{h}$ separately. Thus if we consider the following terms in $\bar{g}$ we have

$$\text{(A7)} \quad \phi_r(\lambda,\theta,t) + \sum_{n=1}^{\infty} \sigma^n \sum_{m=0}^{\infty} [2\frac{\dot{a}}{a} a_{nm} + (m+1)a^{-(m+2)}c_{nm}]P_m$$

$$= a^2\dot{a}[\lambda^{-2} + 2a^{-3}(\lambda - a)]$$

$$+ \sum_{n=1}^{\infty} \sigma^n \sum_{m=0}^{\infty} (m+1)c_{nm}(a^{-(m+2)} - \lambda^{-(m+2)})P_m \in \bar{H}_0^{N+1}.$$

This is proved by noting that since $a \neq 0$, the first term above is in $\bar{H}_0^{N+1}$ according to (e). Furthermore each term in the sum is in $\bar{H}_{n+1}^{N+1}$ by (e), (d) and (a), and therefore in $\bar{H}_0^{N+1}$. Now by (c) the sum is also in $\bar{H}_0^{N+1}$. Thus the statement (A7) is proved. In a similar way the remaining terms of $\bar{g}$, and all the terms in $\bar{h}$ may be shown to belong to $\bar{H}_0^{N+1}$, but the details will be omitted. Therefore (A6) may be considered proved.

Now we consider equations (47) for the a_{nm} and b_n with $n \leq N + 1$. Whenever $m > n$ or $n + m$ = odd, the inhomogeneous terms in the equation for a_{nm} vanish, since these terms are proportional to h_{nm} and g_{nm}, which have just been shown to vanish. The resulting equations are linear and homogeneous, and by (33) the initial data are zero. Therefore it follows that the solutions $a_{nm} = 0$ for $m > n$ or $n + m$ odd. Similarly from equation (47) for b_n we find that $b_n + a_{n1} = 0$ for n even. In order to conclude that $b_n = 0$ for n even we must prove that $a_{n1} = 0$ for n even. Once this is proved, it will also follow from (46) that $c_{nm} = 0$ for $m > n$ or $n + m$ odd and the theorem will have been proved.

Thus we must show that for N odd, $a_{N+1,1} = 0$. To this end we consider equation (27) and expand the integrand in powers of σ, using (31) for λ. The coefficient of σ^{N+1} in this expansion is a sum of terms of the form

$$\text{(A8)} \qquad \int_{-1}^{1} \prod_{\substack{i=1 \\ \Sigma n_i = N+1}}^{4} [a_{n_i m_i} P_{m_i}(x)] x \, dx.$$

In this equation the variable $x = \cos\theta$ has been introduced. If all n_i in a particular term with a non-vanishing integrand are $\leq N$, then since their sum $= N + 1 =$ even, the sum of the corresponding m_i is also even. This follows from the induction hypothesis, since for $n \leq N$, $n_i + m_i =$ even if $a_{n_i m_i} \neq 0$. Consequently $\prod_{i=1}^{4} P_{m_i}(x)$ is an even polynomial in x and therefore the integral vanishes.

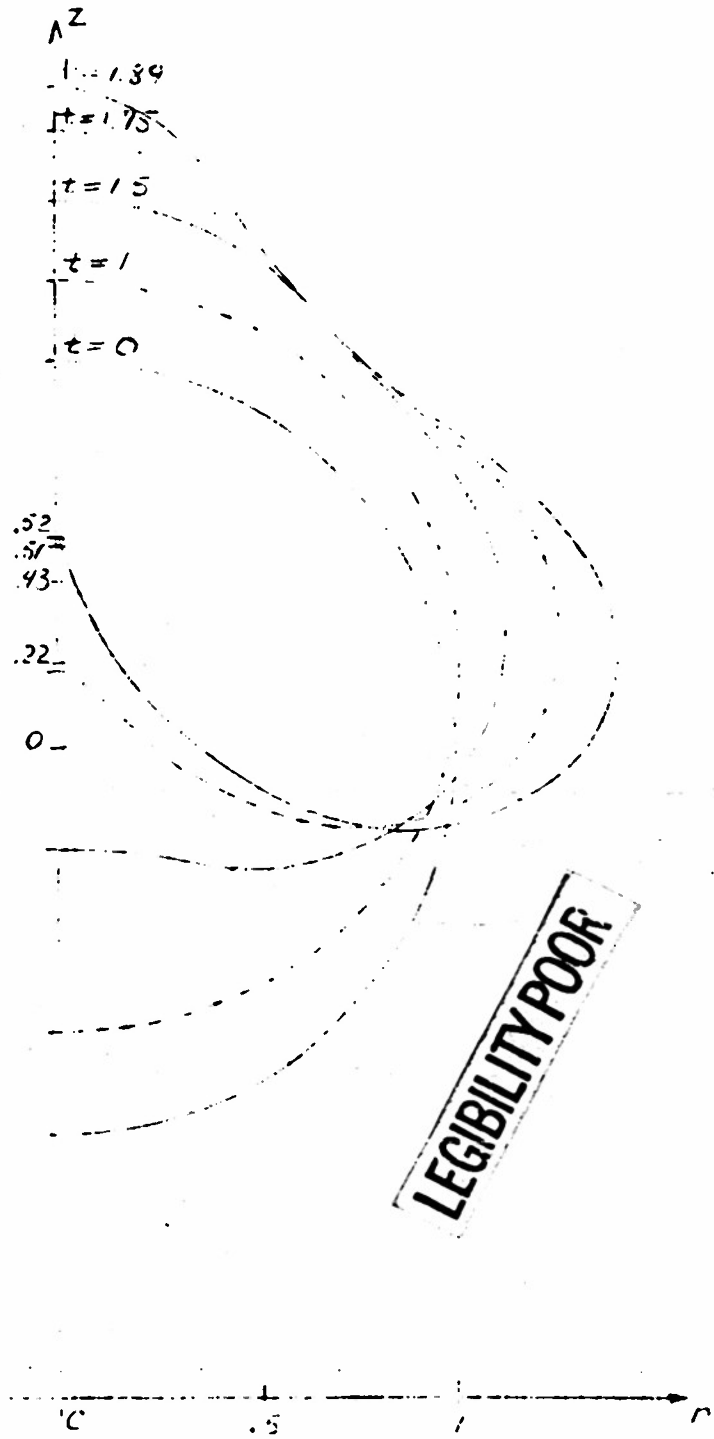
Hence the only terms not vanishing identically are those in which one n, say $n_j = N + 1$ and the other n_i are zero. Now if $n_i = 0$ then $m_i = 0$ since otherwise $a_{om_i} = 0$. Therefore $\prod_{i=1}^{4} P_{m_i}(x) = P_{m_j}(x)$ and the integral vanishes unless $m_j = 1$ due to the orthogonality of Legendre polynomials. Consequently the only non-vanishing terms are of the form

$$\text{(A9)} \qquad a^3 \int_{-1}^{1} a_{N+1,1} P_1(x) x \, dx = \frac{2}{3} a^3 a_{N+1,1}.$$

There are four such terms and thus the coefficient of σ^{N+1} is $\frac{8}{3} a^3 a_{N+1,1}$. From equation (27) this coefficient must vanish and therefore

$$\text{(A10)} \qquad a_{N+1,1} = 0.$$

This completes the proof of the theorem.



$\sigma = .222$

$\delta = 1.25$

$k = \hat{k}$

$a_o = 1$

Figure 1

Rise and Change of Shape of "Equilibrium" Bubble

Profiles are shown for five values of t from t = 0 until t = 1.89, at which time breakdown occurs. The center of gravity is also shown for each value of t by a dash on the z-axis.

References

1. Herring, C. Theory of the pulsations of the gas bubble produced by an underwater explosion. NDRC Division 6 Report C4-SR20 (1941).

2. Ward, A. B. Undex 20 and 40 (1943).

3. Penney, W. G. and Price, A. T. British Report SW-27 (1942).

4. Cole, R. H. Underwater Explosions, Princeton (1948) (contains an account of previous work).

5. Tomotika, S. Instability and Breaking up of a Ring of an Incompressible Perfect Liquid. Proceedings of the Physico-Mathematical Society of Japan, 3rd Series, Vol. 18, No. 10, October 1936.